Question Sheet 6, Differentiation I. v1 2020-21

Verifying the Definition

In these questions we use the Limit Laws to verifying the definition

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}. \tag{1}$$

We will also look at examples where the limit fails to exist so f is not differentiable at a, and where we have to look at both one-sided limits to show that the limit in (1) exists, or not.

1. Using the *definition* of the derivative as a limit, and **not** the differentiation rules, calculate the derivatives of the following functions.

   i) $x^4, \quad x \in \mathbb{R}$ ii) $\sqrt{x}, \quad x > 0$ iii) $\dfrac{1}{1+x^4}, \quad x \in \mathbb{R}.$

2. Recall the results from the Lecture Notes that

   $$\lim_{x \to 0} \frac{\sin x}{x} = 1 \quad \text{and} \quad \lim_{x \to 0} \frac{\cos x - 1}{x} = 0.$$

   Assume the addition formulae for cosine and tangent.

   Prove, by verifying the *definition* that,

   i)
   $$\frac{d}{dx} \cos = -\sin x,$$
   for $x \in \mathbb{R}$,

   ii)
   $$\frac{d}{dx} \tan x = \frac{1}{\cos^2 x}$$
   for $x \notin \left\{ \frac{\pi}{2} + n\pi : n \in \mathbb{Z} \right\}$.

3. Recall the result from the Notes that

$$\lim_{h\to 0} \frac{e^h - 1}{h} = 1.$$

Use this, and the *definition* of derivative, to find the derivatives of

i) e^{2x} ii) xe^x. iii) $\sinh x$.

4. Use the *definition* of derivative to find

$$\frac{d}{dx}\left(e^x \sin x\right)$$

for $x \in \mathbb{R}$.

(You may assume if necessary, that $\sin(a+h) = \sin a \cos h + \cos a \sin h$).

Hint Do not use the result but look at the *proof* of the Product Rule for differentiation and use the idea of "adding in zero".

5. i) Prove that $|\sin\theta|$ is **not** differentiable at $\theta = 0$.

ii) Prove, by verifying the definition, that $|\sin\theta|\sin\theta$ **is** differentiable at $\theta = 0$, and find the value of the derivative.

You may assume that $\lim_{\theta\to 0}\sin\theta = 0$ and $\lim_{\theta\to 0}(\sin\theta)/\theta = 1$.

6. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \dfrac{x^2+4x-12}{x^2-4} & \text{if } x \neq 2, -2 \\ 2 & \text{if } x = 2 \\ 1 & \text{if } x = -2. \end{cases}$$

i) Prove, by verifying the definition, that $f(x)$ is differentiable at $x = 2$, and find the value of the derivative.

ii) Prove that $f(x)$ is not differentiable at $x = -2$.

7. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 2x & \text{if } x \geq 1 \\ x^2 + 1 & \text{if } x < 1 \end{cases} .$$

By verifying the *definition* prove that f is differentiable at $x = 1$ and find the value of the derivative.

8. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x^2 - x & \text{for } x \leq 1 \\ x^3 - 1 & \text{for } x > 1. \end{cases}$$

Prove that f is **not** differentiable at $x = 1$.

(It is quickly seen that the one-sided limits of f at $x = 1$ are both 0, the value of $f(0)$, and so f is continuous at $x = 1$. Thus we have another example that continuous does not imply differentiable.)

9. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x^2 & \text{if } x \geq 0, \\ -x^2 & \text{if } x < 0. \end{cases}$$

i) Use the *definition* to show that f is differentiable at $x = 0$ and find the value of $f'(0)$.

ii) Find $f'(x)$ for **all** $x \in \mathbb{R}$.

iii) Is the derivative f' differentiable on $\mathbb{R}$? Give your reasons.